FORT STREET HIGH SCHOOL

YEAR 12
TRIAL HIGHER SCHOOL CERTIFICATE

2001

MATHEMATICS

EXTENSION 1

Time allowed: 2 Hours
(+ 5 Minutes Reading Time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- The marks allocated for each question are indicated.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- Each new question is to be started on a new page.
- Standard integrals are included.
- If required additional paper may be obtained from the Examination Supervisor on request.

Name :______________________ **Class Teacher:** ______________________

Question No	1	2	3	4	5	6	7	Total	Total
Mark	/12	/12	/12	/12	/12	/12	/12	/84	/100

QUESTION 1

(a) (i) Find $\frac{d}{dx}(x\ln x - x)$ **4**

(ii) Hence evaluate $\int_1^e \ln x\,dx$. Leave the answer in exact form.

(b) Solve the inequality $\frac{x}{x-2} \le 3$. **3**

(c) By using the substitution $u = x^3 + 1$, find $\int x^2\sqrt{x^3+1}\,dx$ **3**

(d) The polynomial $x^3 + 2x^2 + ax + b$ has a factor (x+2) and when divided by (x-2) there is a remainder of 12. Find a and b. **2**

QUESTION 2

(a) (i) Write down the expansion of tan(A+B) **4**

(ii) Find the exact value of $\tan\frac{7\pi}{12}$ in simplest form with rational denominator.

(b) Solve $8\cos^2 x - 8\sin^2 x = 5$ for $0° \le x \le 360°$ **3**

(c) Prove by mathematical Induction that $6^n - 1$ is divisible by 5 for $n \ge 1$ **4**

(d) Given that $\lim_{x\to 0}\frac{\sin x}{x} = 1$, show that $\lim_{x\to 0}\frac{\sin 4x}{9x} = \frac{4}{9}$ **1**

QUESTION 3

(a) A particle moves in a straight line so that its displacement x metres from the origin 0 at the time t seconds is given by $x = 10\sin\frac{t}{2}$ **5**

(i) Show that $\frac{d^2x}{dt^2} = -\frac{x}{4}$

(ii) State the amplitude and the period of the motion.

(iii) Find the maximum speed of the particle.

(b) (i) Show that the normal to the parabola $x^2 = 4ay$ at the point $(2at, at^2)$ has the equation $x + ty = 2at + at^3$ **4**

(ii) Hence show that there is only one normal which passes through its focus.

(c) Find $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x\,dx$ **3**

QUESTION 4

(a) Consider the function $f(x) = 3\sin^{-1} 2x$ **7**

(i) Evaluate $f(\frac{1}{4})$.

(ii) Write down the domain and range of f(x).

(iii) Draw the graph of y=f(x) showing any key features.

(iv) Find the derivative of f(x).

(b) The roots α, β and δ of the equation $2x^3 + 9x^2 - 27x - 54 = 0$ are in geometric progression. **5**

(i) Show $\beta^2 = \alpha\delta$

(ii) Write down the value of $\alpha\beta\delta$.

(iii) Find α, β and δ.

QUESTION 5

(a) The acceleration of a particle is given by $\frac{d^2x}{dt^2} = \frac{-72}{x^2}$ where x metres is the displacement from the origin after 10 seconds. When t=0 the particle is 9 metres to the right of the origin with a velocity of $4m/sec$. **6**

(i) Show the velocity, v, of the particle, in terms of x is $v = \frac{12}{\sqrt{x}}$.

(ii) Find t in terms of x.

(iii) How many seconds does it take for the particle to reach a point 35metres to the right of the origin?

(b) Prove $\frac{\text{cosec}^2 A}{\cot^2 A - 1} = \sec 2A$ $cosec^2 - 1 = cot^2$ **2**

(c) For the function $y = \frac{\pi}{2} - \cos^{-1}(2x)$ **4**

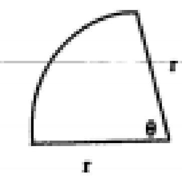
(i) State the domain and range

(ii) Find the value of y when x= 0.25

(iii) Sketch the curve of the function.

QUESTION 6

(a) The diagram below shows the sector of a circle of radius r cm and angle θ radians. The area of the sector is 25 cm^2 **7**



(i) show $\theta = \frac{50}{r^2}$

(ii) If P denotes the perimeter of the sector, show that $P = 2r + \frac{50}{r}$

(iii) Determine the value of r which gives the minimum perimeter

(b) Let T be the temperature inside a room at time t and let A be the constant outside air temperature. Newton's law of cooling states the rate of change of the temperature T is proportional to (T-A). **5**

(i) Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

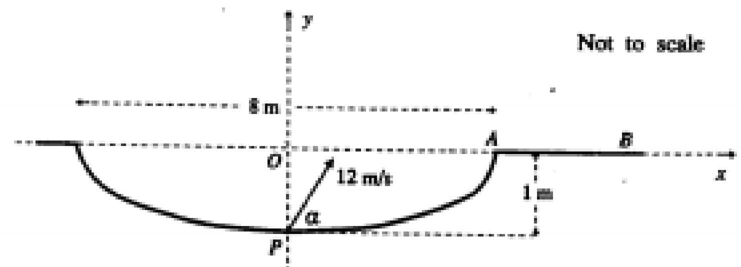
(ii) The out side air temperature is 5°C and a heating system breakdown causes the inside air temperature to drop from 20°C to 17°C in half an hour. After how many hours is the inside room temperature equal to 10°C?

QUESTION 7

(a) Find the maximum value of the function $y = e^{-x}\sin x$, where x is in radians, for the domain $0 \le x \le 2\pi$.(a full explanation is required) **3**

(b) A golf ball is lying at a point P, at the bottom of a bunker, which is surrounded by level ground. The point A is at the edge of the bunker, and the line AB lies on level ground. The bunker is 8 metres wide and 1 metre deep. **9**

The ball is hit towards A with an initial speed of 12 metres per second, and an angle of elevation α. (Have g=10 $\frac{m}{s^2}$)



(i) Show that the golf ball's trajectory at time t seconds after being hit can be defined by the equations

$x = (12\cos\alpha)t$ and $y = -5t^2 + (12\sin\alpha)t - 1$

Where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O as shown in the diagram.

(ii) Given α = 30°, how far from A will the ball land?

(iii) Find the maximum height the level groung reached by the ball if α = 30°.

(iv) Find the range of values of α, to the nearest degree, at which the ball must be hit so it will land to the right of A.

ntions Ext. 1 Mathematics 2001 trial. | Comments

) $y = x\ln x - x$, $y' = x \cdot \frac{1}{x} + \ln x \cdot 1 - 1$ ✓✓

$= 1 + \ln x - 1 = \ln x$

) $\int_2^e \ln x \, dx = x\ln x - x \Big]_2^e = (e\ln e - e) - (2\ln 2 - 2)$

$= 2(1 - \ln 2)$ ✓✓

Q1
(a)(i) Many students need to practice product rule as they made careless errors.

$\frac{x}{x-2} \le 3 \quad [\times (x-2)^2] \quad x(x-2) \le 3(x-2)^2$ ✓

$x^2 - 2x \le 3(x^2 - 4x + 4)$, $x^2 - 2x \le 3x^2 - 12x + 12$ ✓

$0 \le 2x^2 - 10x + 12$, $0 \le 2(x-3)(x-2)$  ✓

$\therefore \quad x < 2$ or $x \ge 3 \quad (x \ne 2)$

b) Memorize this method It is the easiest to use

$u = x^3 + 1 \quad \therefore \frac{du}{dx} = 3x^2 \quad \therefore dx = \frac{du}{3x^2}$ ✓

$\int x^2 \sqrt{x^3+1}\, dx = \int x^2 \sqrt{u} \, \frac{du}{3x^2} = \frac{1}{3}\int u^{1/2} du$ ✓

$= \frac{1}{3}\left[\frac{2u^{3/2}}{3} + c\right] = \frac{2\sqrt{(x^3+1)^3}}{9} + c$ ✓

c) $\frac{1}{2}$ mark off for not replacing u with x^3+1 at end of working out.

when $x = -2$ $\quad (-2)^3 + 2(-2)^2 + a(-2) + b = 0$

$-8 + 8 - 2a + b = 0$ or $-2a + b = 0$ (1)

when $x = 2$ $\quad (2)^3 + 2(2)^2 + a(2) + b = 12$ ✓

$8 + 8 + 2a + b = 12$ or $2a + b = -4$ (2)

solving (1) and (2) $\quad 2b = -4 \quad \therefore b = -2$

$a = -1$ ✓

i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ ✓

i) $7\pi/12 = \pi/4 + \pi/3 \quad \tan \pi/4 = 1, \ \tan \pi/3 = \sqrt{3}$ ✓

$\tan(\pi/4 + \pi/3) = \dfrac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$ ✓

$= \dfrac{1 + 2\sqrt{3} + 3}{1 - 3} = \dfrac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$ ✓

$8\cos^2 x - 8\sin^2 x = 5, \quad \cos^2 x - \sin^2 x = 5/8$ ✓

$\cos 2x = 5/8 \ \therefore 2x = 51°19'$, ✓ ✓

$x = 25°40', 154°20', 205°40', 334°20'$

for $n=1 \quad 6^1 - 5 = 5 \ \therefore$ true for $n=1$

ssume true for $n=k$ ie $6^k - 1 = 5m$ where m ✓
a positive integer.

ve true for $n=k+1$

$6^{k+1} - 1 = 6 \times 6^k - 6 + 6 - 1 = 6(6^k - 1) + 5$ ✓

$= 6(5m) + 5 = 5(6m+1) \ \therefore$ divisible by 5 ✓

ie true for $n=1$ and for the next value of n ✓
$n=2$ and so on it is true for all values of $n \geq 1$

$\lim_{x\to 0} \dfrac{\sin 4x}{9x} = \lim_{x\to 0} \dfrac{\sin 4x}{9/4(4x)} = \lim_{x\to 0} \dfrac{4}{9} \times \dfrac{\sin 4x}{4x} = 4/9$

✓

i) $x = 10\sin t/2$, $\dot{x} = 10\left(\frac{1}{2}\right)\cos t/2$

$\ddot{x} = -\left(\frac{1}{2}\right)^2\left(10\sin t/2\right) = -\frac{1}{4}x = -x/4$ ✓✓

ii) amplitude $= 10$ metres

Period $= 2\pi/\frac{1}{2} = 4\pi$ ✓

ii) $\dot{x} = 5\cos t/2$ max speed when $\cos t/2 = \pm 1$

$\therefore = 5$ metres/sec. ✓✓

(i) $y = x^2/4a$ $y' = 2x/4a = \frac{x}{2a}$, when $x = 2at$

$y' = M_{tangent} = 2at/2a = t$ $\therefore M_{Normal} = -1/t$ ✓✓

using $y - y_1 = m(x - x_1)$ $y - at^2 = -\frac{1}{t}(x - 2at)$

$\therefore yt - at^3 = -x + 2at$ $\therefore x + ty = 2at + at^3$

ii) If Normal goes through $(0, a)$

$0 + t(a) = 2at + at^3$, $0 = at + at^3$, $0 = at(1 + t^2)$

this has only one solution for t, $t = 0$ (since $1 + t^2 = 0$ has no solution) ✓✓

$\int_0^{\pi/2} \cos x\,[\sin x]^3\,dx$, if $u = \sin x$ $du/dx = \cos x$

$\therefore dx = du/\cos x$ ✓

$\int \cos x\, u^3\, du/\cos x = \int u^3 dx = u^4/4 = \left[\sin x\right]^4/4\Big]_0^{\pi/2}$ ✓

$= \frac{\left[\sin \frac{\pi}{2}\right]^4 - \left[\sin 0\right]^4}{4} = \frac{1}{4}$ ✓

i) Generally well done. Important to use calculus rather than quo SHM equations.

ii) Some mechani errors here, esp. finding the perio

iii) A few differen methods employe here. Some studen lost marks for not taking absolute value.

(b) (i) very well done by most students.

ii) relatively poo response here

(c) (i) few problem encountered here if correc substitution was used.

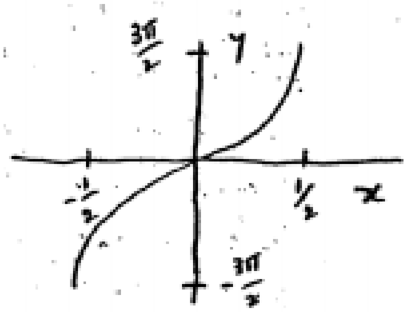
LBeevers.

) $f(\frac{1}{4}) = 3\sin^{-1}\frac{1}{2} = 3(\frac{\pi}{6}) = \frac{\pi}{2}$ ✓

) $-1 \le 2x \le 1 \quad \therefore$ domain $-\frac{1}{2} \le x \le \frac{1}{2}$

Range $y = \sin^{-1}\theta$ is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ✓

$\therefore$ range $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ ✓



✓ ✓

$f'(x) = 3 \times \dfrac{2}{\sqrt{1-(2x)^2}} = \dfrac{6}{\sqrt{1-4x^2}}$ ✓✓

i) If a G.P. $\beta/\alpha = \delta/\beta \quad \therefore \beta^2 = \alpha\delta$ ✓

ii) $\alpha\beta\delta = 54/2 = 27$ ✓

) $\beta^2 = \alpha\delta \quad \therefore \beta^2 \times \beta = \beta^3 = 27 \quad \therefore \beta = 3$ ✓

$\therefore \alpha\delta = 9$

Also $\alpha + \beta + \delta = -9/2 \quad \therefore \alpha + 3 + 9/\alpha = -\frac{9}{2}$

$\therefore 2\alpha^2 + 15\alpha + 18 = 0$

$(2\alpha + 3)(\alpha + 6) = 0$ ✓

$\therefore \alpha = -6$ or $\alpha = -3/2$

$\delta = -3/2$ or $\delta = -6$ ✓

$\therefore \alpha, \beta, \delta$ are $-6, 3, -3/2$

Q4

(a)

(ii) If domain and range incorrect but graph is ok 2/4

(iv) Best to use $y = \sin^{-1} f(x) \quad y' = \dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$

(iv) If the student did not use $\beta^2 = \alpha\delta$ then the question is difficult to '-

(i) $\dfrac{d^2x}{dt^2} = \dfrac{-72}{x^2}$

$acc. = \dfrac{d\,\frac{1}{2}v^2}{dx} = -72x^{-2}$

$\therefore \frac{1}{2}v^2 = \int -72x^{-2}\,dx = \dfrac{-72x^{-1}}{-1} + C = \dfrac{72}{x} + C$

when $x = 9$, $v = 4$ $\quad \frac{1}{2}(4)^2 = \frac{72}{9} + C$

$\therefore C = 0 \quad \therefore \frac{1}{2}v^2 = \frac{72}{x} \quad v^2 = \frac{144}{x}$

$v = \pm \frac{12}{\sqrt{x}}$. Since when $t = 0$ v is $+$ $\therefore$ use

$v = \frac{12}{\sqrt{x}}$ ✓✓✓

$\frac{dx}{dt} = v = \frac{12}{\sqrt{x}} \quad \therefore \frac{dt}{dx} = \frac{x^{1/2}}{12}$

$\therefore t = \int \frac{x^{1/2}}{12}\,dx = \dfrac{2x^{3/2}}{3 \times 12} + C = \dfrac{\sqrt{x^3}}{18} + C$

when $t = 0$, $x = 9$ $\therefore 0 = \frac{27}{18} + C \therefore C = -\frac{27}{18} = -\frac{3}{2}$

$\therefore t = \frac{\sqrt{x^3}}{18} - \frac{3}{2}$ ✓✓

) $t = \frac{\sqrt{35^3}}{18} - \frac{3}{2} \doteqdot 10$ seconds ✓

$\dfrac{\operatorname{cosec}^2 A}{\cot^2 A - 1} \equiv \sec 2A$ ✓✓

$LHS = \dfrac{1/\sin^2 A}{\cos^2 A/\sin^2 A - 1} = \dfrac{1/\sin^2 A}{\frac{\cos^2 A - \sin^2 A}{\sin^2 A}} = \dfrac{1}{\cos^2 A - \sin^2 A} = \dfrac{1}{\cos 2A} = \sec 2A$

$\therefore LHS = RHS$

(i) Domain $-1 \le 2x \le 1 \quad \therefore -\frac{1}{2} \le x \le \frac{1}{2}$

Range of $y = \cos^{-1}\theta$ is $0 \le \theta \le \pi \quad \therefore \frac{\pi}{2} - 0 = \frac{\pi}{2}, \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$\therefore$ Range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ✓✓

ii) $y = \frac{\pi}{2} - \cos^{-1}\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ✓

ii)

✓

$y' = e^{-x}\cos x + e^{-x}(-1)\sin x$ for St. points

$0 = e^{-x}\cos x - e^{-x}\sin x = e^{-x}(\cos x - \sin x)$

$e^{-x} \neq 0$ $\therefore$ only solutions $\cos x - \sin x = 0$ ✓

- y'
- solns

$\cos x = \sin x$ $\therefore$ $\tan x = 1$ $\quad x = \pi/4,\ 5\pi/4$ ✓

- testing it is a max.
- testing for turning points ✓

$x = \pi/4$

x	0.7	$\pi/4$	0.8
y'	+	0	−

$\therefore$ when $x = \pi/4$ this is a maximum turning point

$x = \frac{5\pi}{4}$

x	3.5	$\frac{5\pi}{4}$	4
y'	−	0	+

$\therefore$ when $x = 5\pi/4$ this is a minimum turning point

$\therefore$ Max value $y = e^{-\pi/4}\sin\pi/4 \doteq 0.32$

. most students thought that the max would occur when sin x was a max.

1 $\dot{x} = 12\cos\alpha$, $\quad \dot{y} = 12\sin\alpha$ at $t = 0$

$x = 0 \qquad y = -1$

$\ddot{x} = 0 \qquad \ddot{y} = -10$

$x = (12\cos\alpha)t \qquad acc = \ddot{y} = dv/dt = -10$ ✓✓

- eqns
- derivation

$\therefore V = \dot{y} = \int -10\,dt = -10t + c$

$c = 12\sin\alpha$ $\therefore V = \frac{dy}{dt} = 12\sin\alpha - 10t$

$y = \int 12\sin\alpha - 10t\,dt = (12\sin\alpha)t - 5t^2 + c$

when $t = 0$ $y = -1$ $\therefore c = -1$

$\therefore y = -5t^2 + (12\sin\alpha)t - 1$

. a number of students quoted the eqns rather than deriving them.

Ball lands when $y = 0$, $\sin 30 = 0.5$

$0 = -5t^2 + 6t - 1 \qquad (5t-1)(t-1) = 0$, $t = \frac{1}{5}$ and 1

$\therefore x = (12\cos 30°)(1) \doteq 10.39$

$\therefore$ distance from A $\doteq 10.39 - 4 = 6.39$ m

$6\sqrt{3} - 4$

- x
- distance from A ✓

. students are not solving quadratic eqn correctly.

. students are not reading the question to see what is required but just quoting formulae which requires some rele to question eg in this part - wrong range.

(cont)

Max height when $\dot{y}=0$ or t or height

$\therefore 0 = -10t + 12\sin 30°$

$0 = -10t + 6$

$\therefore t = 0.6$ sec

$y = -5t^2 + (12\sin\alpha)t - 1 \quad \therefore y = -5(0.6)^2 + 6(0.6) - 1$

$= 0.8\text{m}$ ✓✓

Point A is $(4,0)$

when $x=4 \quad 4 = (12\cos\alpha)t \quad \therefore t = \dfrac{1}{3\cos\alpha}$ ✓ ①

when $y=0 \quad 0 = -5t^2 + (12\sin\alpha)t - 1$ ②

Sub ① into ②

$$-5\left(\frac{1}{3\cos\alpha}\right)^2 + 12\sin\alpha \times \frac{1}{3\cos\alpha} - 1 = 0$$

$$5\sec^2\alpha - 36\tan\alpha + 9 = 0 \quad (\sec^2\alpha = \tan^2\alpha + 1)$$

$$\therefore 5\tan^2\alpha - 36\tan\alpha + 14 = 0$$

$$\tan\alpha = 36 \pm \sqrt{36^2 - 4(5)(14)} / 10 \doteqdot 0.4125 \text{ or } 6.7875$$

$$\therefore \alpha = 22.4 \text{ or } 81.6$$

since when rounding to nearest degree $\alpha \neq 22°$ or $82°$

$\therefore 23° \leq \alpha \leq 81°$ ✓✓✓

1 for $t = \frac{1}{3\cos\alpha}$

1 for a quad.

1 for correct angles.

• most students did not know what to do with this question

• those that formed the quadratic could not solve it correctly.